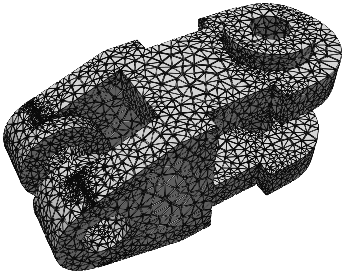
Fast Hexahedral Mesh Extraction from Locally Injective Integer-Grid Maps

Tobias Kohler

October 5, 2023

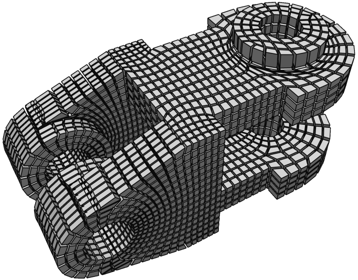
Motivation

Tet-Mesh



- easier to generate

Hex-Mesh



- nicer numerical features

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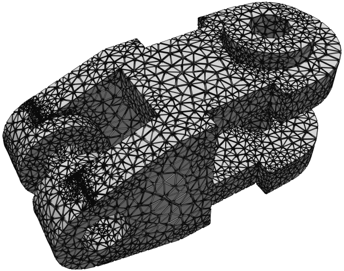

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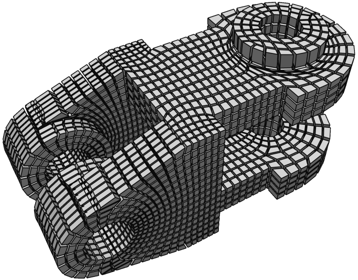

- nicer numerical features

Integer-Grid Map (IGM)

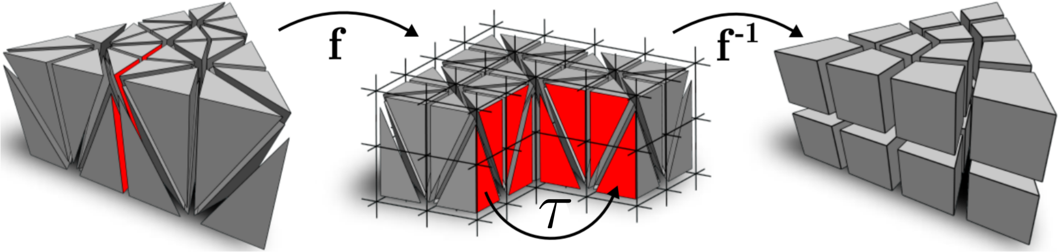

Image Source: (Lyon et al. 2016)

Remark

parametrization f is per cell and transition τ is per half-face.

Robust HexEx (Lyon et al. 2016)

- Extracts a hex-mesh from a given tet-mesh and an IGM.
  1. Preprocessing - extract $\boldsymbol{\tau}$ from $\boldsymbol{f}$ and resolve floating-point inaccuracies.

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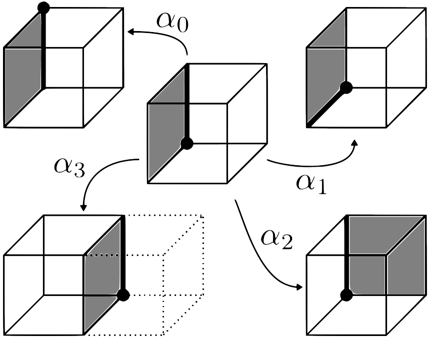
  2. Geometry Extraction - extract a hex-vertex for each intersection of the parametrization with the integer-grid.
  3. Topology Extraction - enumerate darts and trace their connections through the parametrization.
  4. Postprocessing - resolve problems due to flipped or degenerate cells.

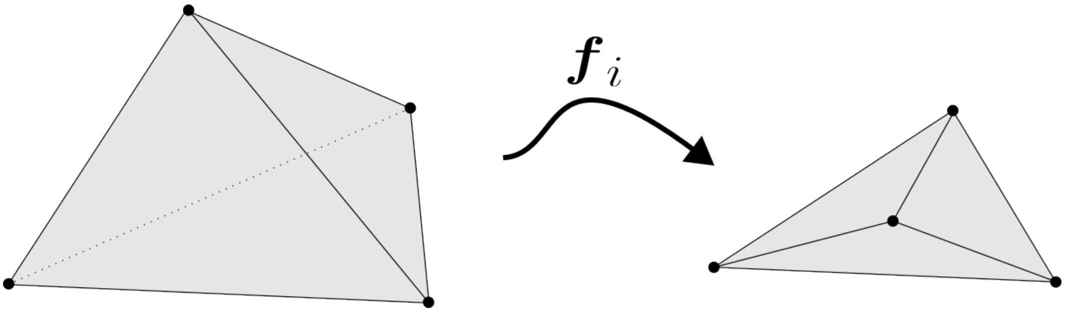
Robust HexEx (Lyon et al. 2016)

Darts (Kraemer et al. 2014)

$6 \cdot 4 \cdot 2 \cdot |C| = 48|C|$ darts
$4 \cdot 48|C| = 192|C|$ connections



Degenerate Cells



No local injectivity!

Robust HexEx - Room for Improvements

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- Geometry Extraction: All points in the bounding box are tested
- Topology Extraction: Inefficient data structure
- General: Potentially long lists are searched in linearly

From Robust HexEx to Fast HexEx

Goal: Make it faster!

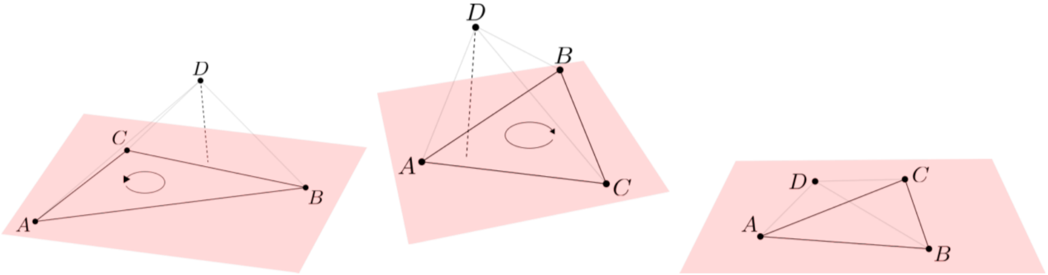
Fast HexEx (Ours)

Extracts a hex-mesh from a given tet-mesh and an IGM. Local injectivity is required.

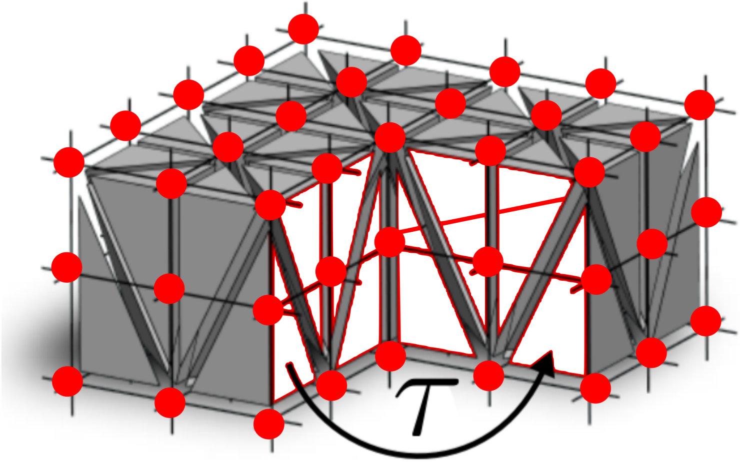
1. Preprocessing - unchanged
2. Geometry Extraction - using rasterization
3. Topology Extraction - with propellers
4. ~~Postprocessing~~

Exact Predicates (Shewchuk 1996)

All geometric tests like ONLINESEGMENT, INTRIANGLE, … build on ORI2D and ORI3D.

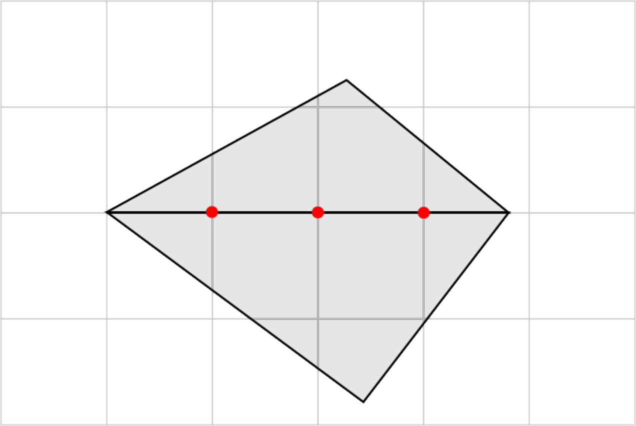


Hex-Vertex Extraction



Hex-Vertex Extraction

Duplicates



Hex-Vertex Extraction

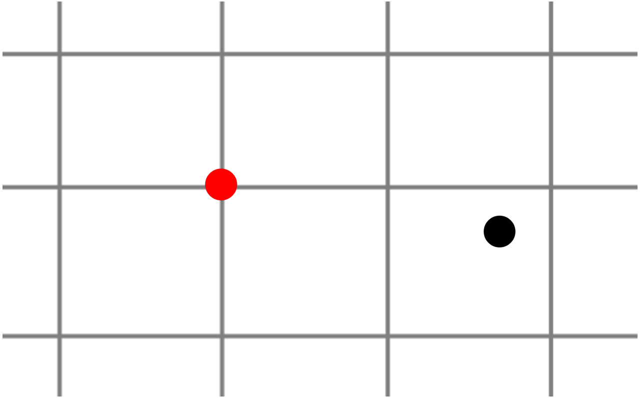
Algorithm Vertex Extraction on Edges (High-Level)

1: **for** $e \in E$ **do**
2: pick any cell $c \in C$ s.t. $e \sim c$
3: $Z \leftarrow \mathring{\boldsymbol{F}}_c(e) \cap \mathbb{Z}^3$
4: **if** $Z \neq \emptyset$ **then**
5: $G \leftarrow G \cup \{e\}$
6: **for** $\boldsymbol{z} \in Z$ **do**
7: generate hex-vertex with generator e, geometric embedding $\boldsymbol{f}_c{}^{-1}(\boldsymbol{z})$
8: and parameter $\boldsymbol{z}$ in the chart of c

Goal: Find $\mathring{\boldsymbol{F}}_c(x) \cap \mathbb{Z}^3$ efficiently.

The Easy Case - Vertices

$$\{(x,\ y,\ z)\} \cap \mathbb{Z}^3$$



Hex-Vertex Extraction - Candidates

- Which points to test against the exact predicates?

Hex-Vertex Extraction - Candidates

- All of $\mathbb{Z}^3 \Rightarrow$ infeasible

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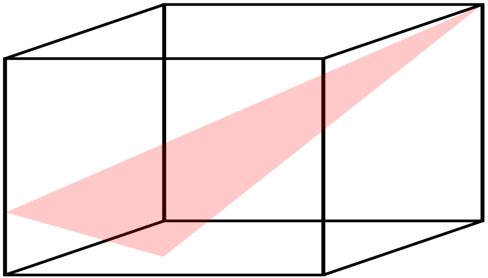
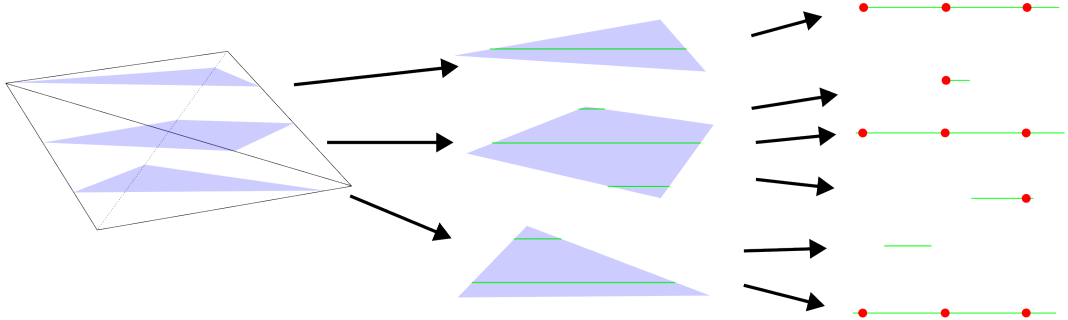
- Bounding box $\Rightarrow$ inefficient



Hex-Vertex Extraction - Candidates

- All of $\mathbb{Z}^3 \Rightarrow$ infeasible
- Bounding box $\Rightarrow$ inefficient
- Rasterize element $\Rightarrow$ keep exact predicate calls to a minimum

Rasterization - Top-Down

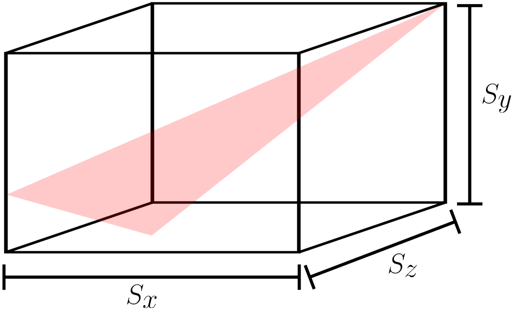


Rasterization - Coordinate Permutation

For simplicity and efficiency:

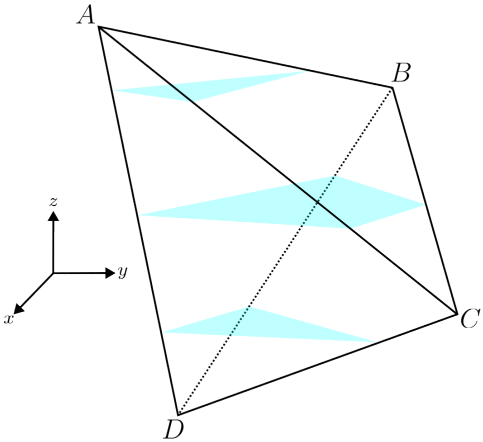
$$S_x \geq S_y \geq S_z$$

Rasterization axis $r := \arg\min_i(S_i > 0)$



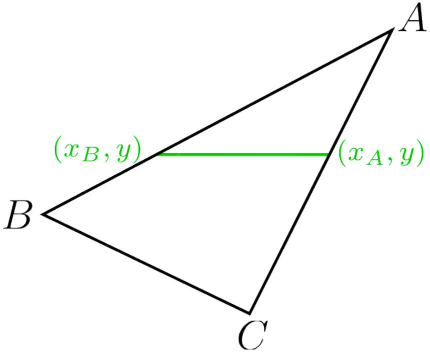
Rasterizing Cells/Tetrahedra

$A_z \geq B_z \geq C_z \geq D_z$



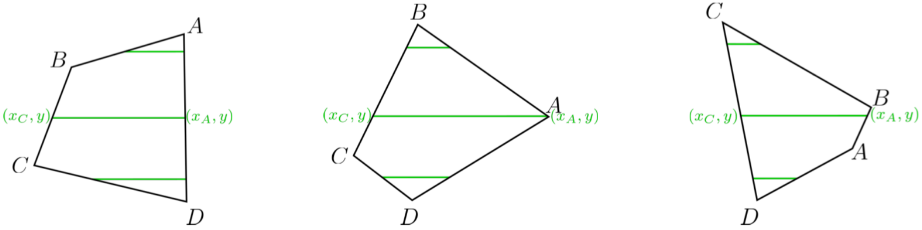
Rasterizing Faces/Triangles

$A_r \geq B_r \geq C_r$

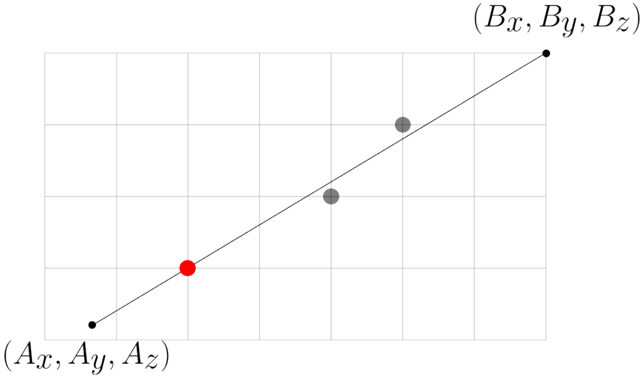


Rasterizing Quads

$A_r, B_r, C_r \geq D_r \Rightarrow$ 3 cases

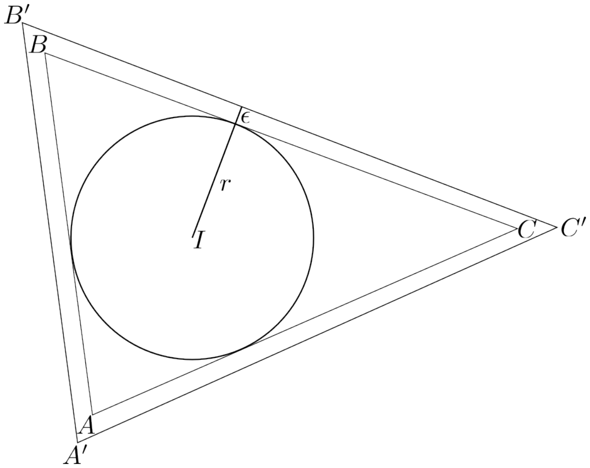


Rasterizing Edges/Line Segments



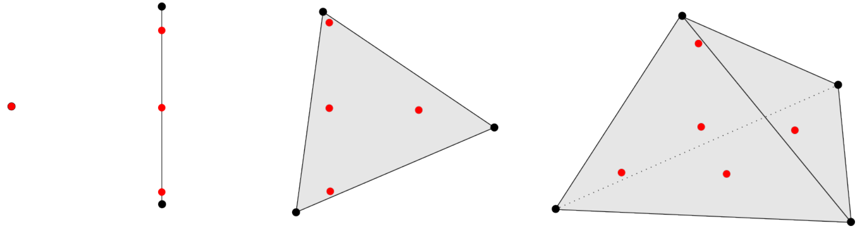
Trivial case: $S_y = S_z = 0$

Floating-Point Inaccuracies



Hex-Vertex Extraction

- Result: List of generators, each with list of isolated hex-vertices



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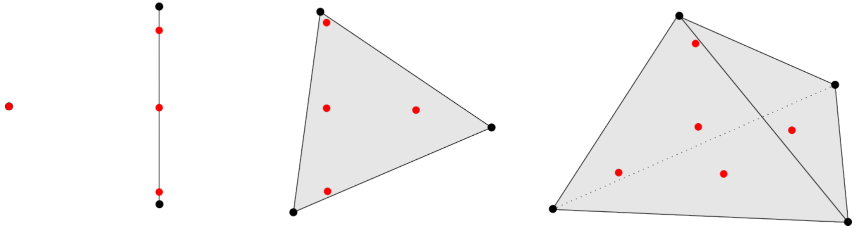
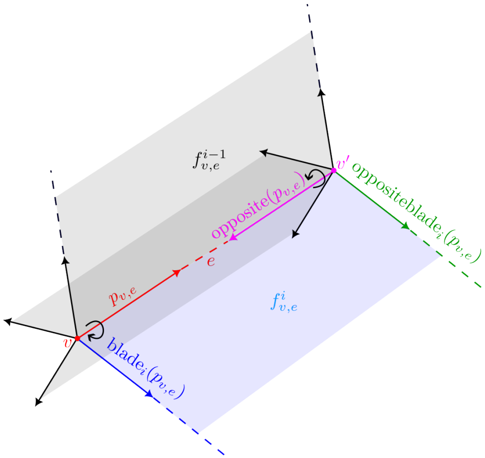


- Missing: Connectivity (hex-edges, -faces, -cells)

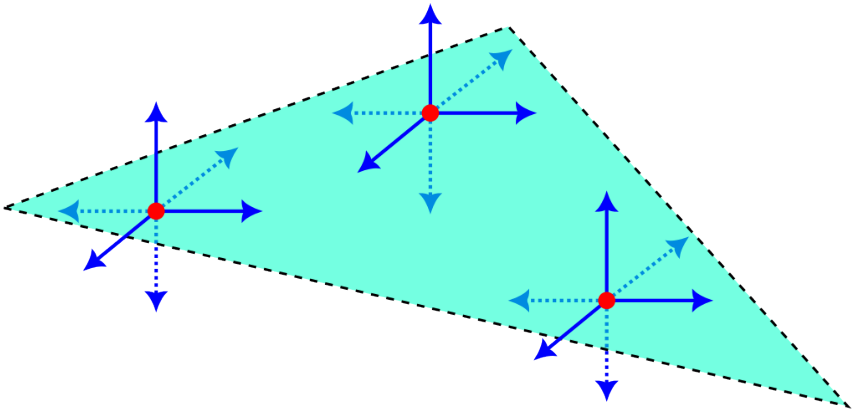
Propellers

Propellers

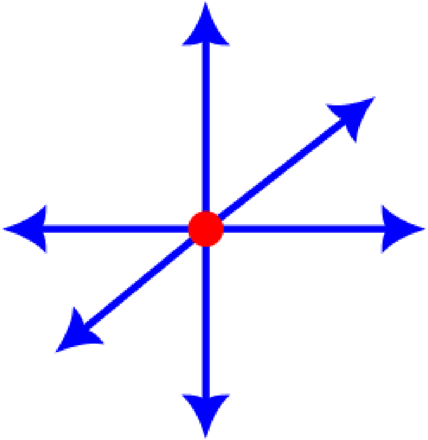
$2|E|$ propellers
$2|E| + 8|F|$ connections



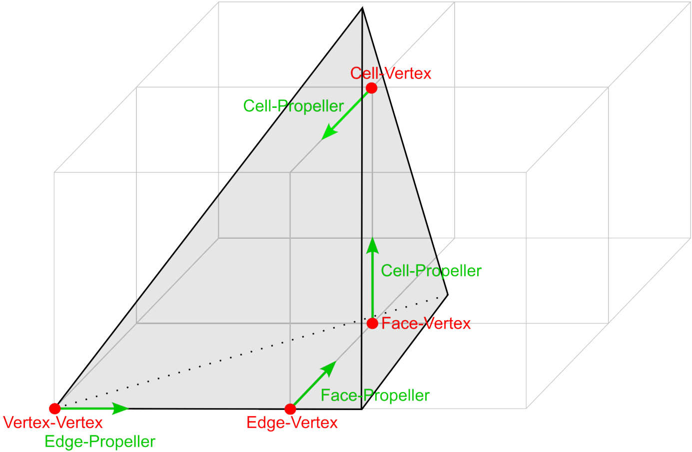
Local Topology per Generator



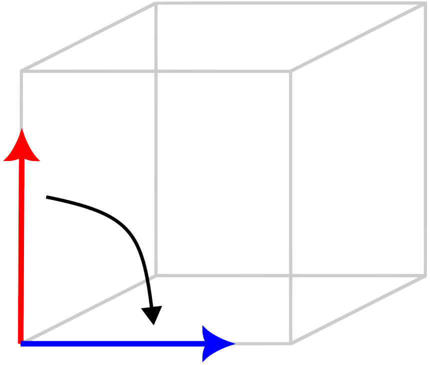
Local Topology in Cells

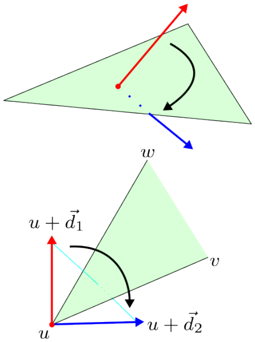


Local Topology - Propeller Roots

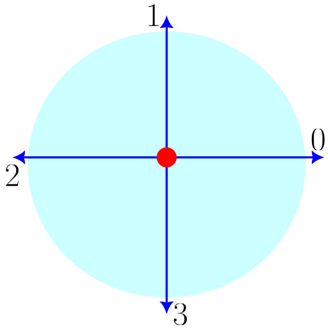


Local Topology - Rotating from Roots to Blades



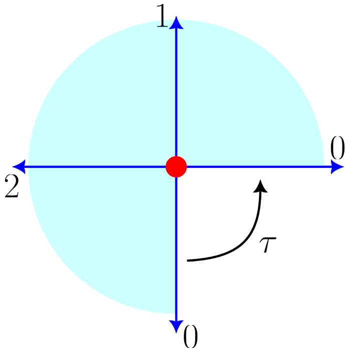


Regular Hex-Edges

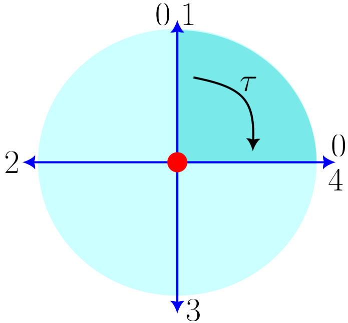


Valence 4 inner regular edge

Singular Hex-Edges



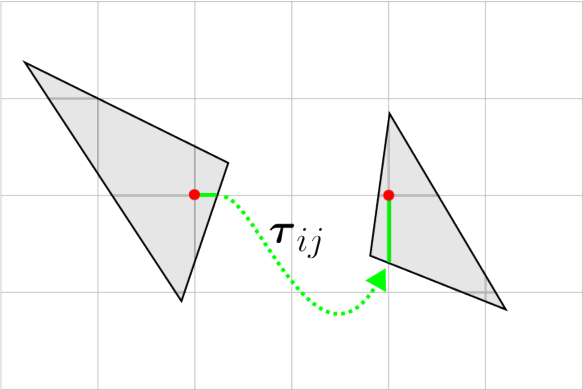
Valence 3 inner singularity edge



Valence 5 inner singularity edge

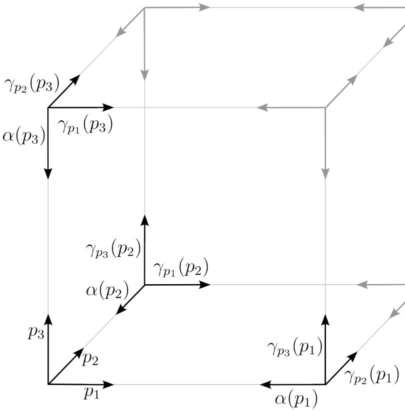
Tracing

"Connecting the dots"



The Final Step - Hex-Cells

3 propellers make up a hex-corner, 8 hex-corners define a hex-cell.



The Algorithm

preprocessing()
for $x \in V \cup E \cup F \cup C$ **do**
    extractHexVertices(x)
for $g \in G \setminus C$ **do**
    enumeratePropellerRoots(g)
    for $p \in \mathcal{P}_g$ **do**
        connectPropellerBlades(p)
for $v_h \in V_h$ **do**
    $g \leftarrow$ generator of v_h
    for $p \in \mathcal{P}_g$ **do**
        connectPropellerOpposite(v_h, p)
extractHexCells()

Improvements to Robust HexEx

- Rasterization instead of bounding box check

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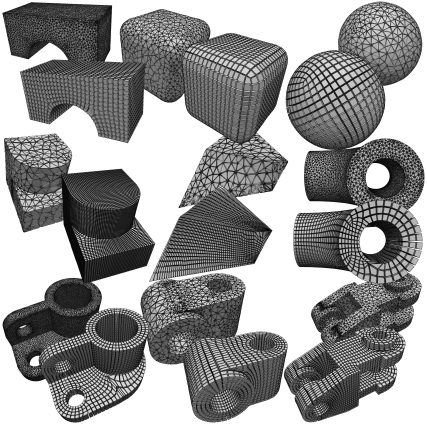
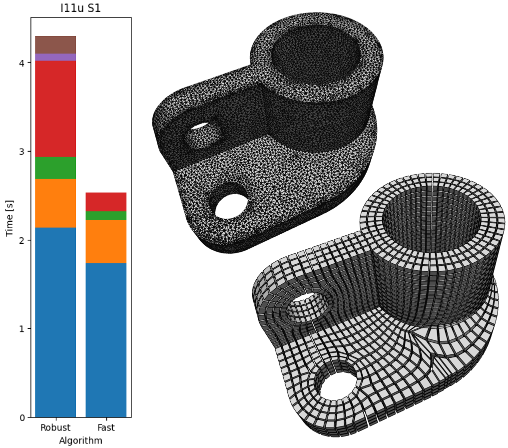
- Propellers instead of darts
- Local topology is stored per generator instead of per hex-vertex
- Hash-maps for potentially large collections instead of lists

Results



HexMe dataset (Beaufort et al. 2022)

A Coarse Example

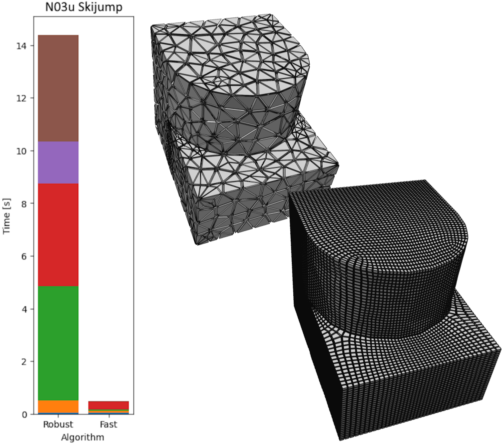


$$\frac{H}{T} = 2\%$$

$$\frac{\text{Fast}}{\text{Robust}} = 59\%$$

- Preprocessing
- Vertex Extraction
- Local Connections
- Tracing Connections
- Postprocessing
- Cell Extraction

A Fine Example



$$\frac{H}{T} = 1175\%$$

$$\frac{\text{Fast}}{\text{Robust}} = 3\%$$

- Preprocessing
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- Tracing Connections
- Postprocessing
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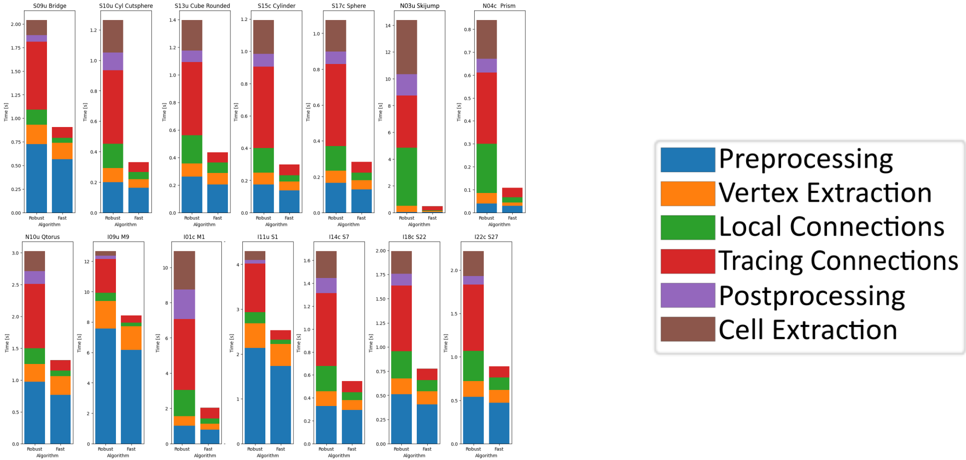
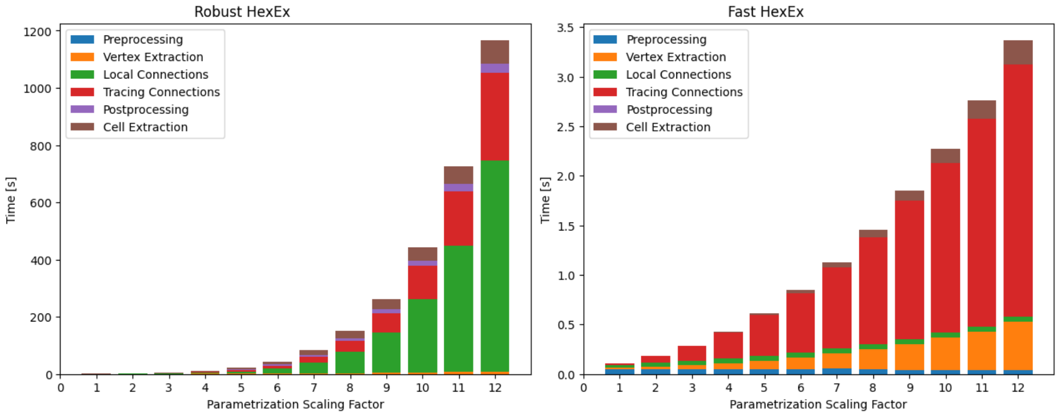
Timings

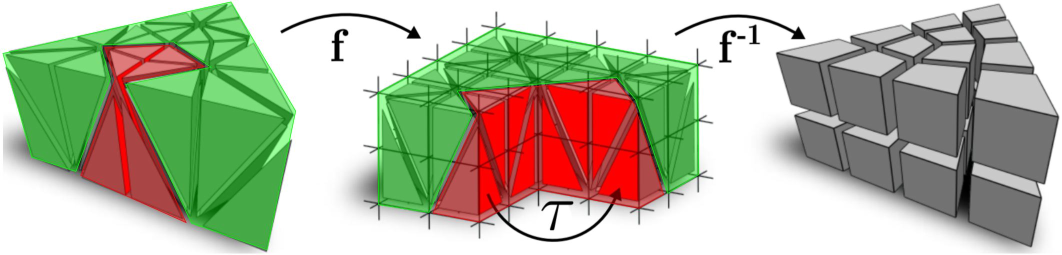

S09u Bridge
S10u Cyl Cutsphere
S13u Cube Rounded
S15c Cylinder
S17c Sphere
N03u Skijump
N04c Prism
N10u Qtorus
I09u M9
I01c M1
I11u S1
I14c S7
I18c S22
I22c S27
Time [s]
Robust
Fast
Algorithm
Preprocessing
Vertex Extraction
Local Connections
Tracing Connections
Postprocessing
Cell Extraction

Scaling Comparison



Potential Improvements

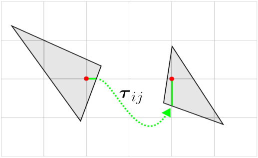
Operate on entire blocks of cells.

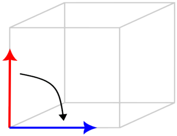


Potential Improvements

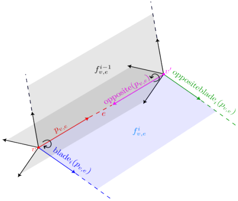
Parallelization (Vertex Extraction, Local Topology, ...)

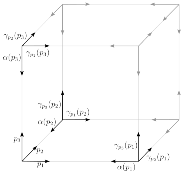
Thank You

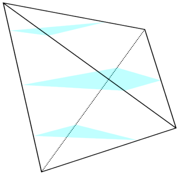




Questions?







References

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